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ON HARADA RINGS. II

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

KIYOICHI OSHIRO

Let R be a left artinian ring and E a complete set of orthogonal primitive idempotents. R is said to be a left Harada (H -) ring if E is arranged as

$$E = \{e_{11}, \dots, e_{1n(1)}, e_{21}, \dots, e_{2n(2)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

for which

- 1) each $e_{i1}R$ is an injective right R -module,
- 2) $e_{i,j-1}R \simeq e_{ij}R$ or $e_{i,j-1}J \simeq e_{ij}R$ for $1 \leq i \leq m$, $1 < j \leq n(i)$

where J is the Jacobson radical of R .

In [4] (cf. [3]), we showed that, for a left artinian ring R , the following conditions are equivalent :

- 1) R is a left Harada ring.
- 2) The family of all injective left R -modules is closed under taking small covers.
- 3) The family of all projective right R -modules is closed under taking essential extensions.

In the present paper, we shall show that left H -rings can be represented as suitable generalized matrix rings. As left H -rings are Morita invariant, we may restrict our attention to basic left H -rings.

Notation. Throughout this paper, rings R considered are associative rings with identity and all R -modules are unitary. The notation M_R (resp. M_R) is used to stress that M is a right (resp. left) R -module. For an R -module M , $J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively, and $\{J_i(M)\}$ and $\{S_i(M)\}$ denote its descending Loewy chain and ascending Loewy chain, respectively.

For R -modules M and N , for the sake of convenience, we put

$$(M, N) = \text{Hom}_R(M, N)$$

and in particular, put

$$(e, f) = (eR, fR) = \text{Hom}_R(eR, fR)$$

Now, henceforth, we assume that R is a basic H -ring and E a com-

plete set of orthogonal primitive idempotents of R . So, E is arranged as

$$E = \{e_{11}, \dots, e_{1n(1)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

for which

- 1) each $e_{i1}R_R$ is injective,
- 2) there exists an isomorphism $\theta_{k,k-1}^i$ from $e_{ik}R_R$ to $e_{i,k-1}J(R)_R = J(e_{i,k-1}R_R)$.

We put

$$\begin{aligned}\theta_{1,1}^i &= \text{the identity map of } e_{i1}R_R \\ \theta_{k,1}^i &= \theta_{2,1}^i \cdots \theta_{k-1,k-2}^i \theta_{k,k-1}^i\end{aligned}$$

We represent R as

$$\begin{aligned}R &= \begin{pmatrix} (e_{11}, e_{11}) & \cdots & (e_{mn(m)}, e_{11}) \\ & \ddots & \\ (e_{11}, e_{mn(m)}) & \cdots & (e_{mn(m)}, e_{mn(m)}) \end{pmatrix} \\ &= \begin{pmatrix} e_{11}Re_{11} & \cdots & e_{11}Re_{mn(m)} \\ & \ddots & \\ e_{mn(m)}Re_{11} & \cdots & e_{mn(m)}Re_{mn(m)} \end{pmatrix}\end{aligned}$$

The following properties hold on R ([4]) :

- a) Each $S(e_{ij}R_R)_R$ is simple,

$$\begin{aligned}S(e_{i1}R_R)_R &\simeq \cdots \simeq S(e_{in(i)}R_R)_R, \\ S(e_{ij}R_R)_R &\neq S(e_{kt}R_R)_R \text{ if } i \neq k,\end{aligned}$$

- b) For each e_{i1} , there exists unique $g_i \in E$ such that $(e_{i1}R_R; {}_R Rg_i)$ is an injective pair, i.e., $g_iR_R/J(g_iR_R)_R \simeq S(e_{i1}R_R)_R$ and ${}_R Re_{i1}/J({}_R Re_{i1}) \simeq {}_R S({}_R Rg_i)$. Then ${}_R Rg_i$ is injective, and

$$S_k({}_R Rg_i) = S(e_{i1}R_R) + \cdots + S(e_{ik}R_R)$$

for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. So, $S_k(Rg_i)$ is a two-sided ideal. In particular, $S({}_R Rg_i) = S(e_{i1}R_R)$ is a simple ideal. In the matrix representation,

$$S_k({}_R Rg_i) = \begin{pmatrix} & 0 & & \\ & \vdots & & \\ & 0 & & \\ 0 & X_1 & & 0 \\ & \vdots & & \\ & X_k & & \\ & 0 & & \\ & \vdots & & \\ & 0 & & \end{pmatrix}$$

where $X_j = S(e_{ij} Re_{ij} e_{ij} R g_i) = S(e_{ij} R g_i g_i R g_i)$ for $1 \leq j \leq k$.

Here we define two mappings :

$$\begin{aligned}\sigma &: \{1, \dots, m\} \rightarrow \{1, \dots, m\} \\ \rho &: \{1, \dots, m\} \rightarrow \{1, \dots, n(1)\} \cup \dots \cup \{1, \dots, n(m)\}\end{aligned}$$

by the rule $\sigma(i) = k$ and $\rho(i) = t$ if $g_i = e_{kt}$; namely $(e_{i1} R_R ; {}_R Re_{\sigma(i), \rho(i)})$ is an injective pair. We note that $\{\sigma(1), \dots, \sigma(m)\} \subseteq \{1, \dots, m\}$ and $\rho(i) \leq n(\sigma(i))$.

We define R_{ij} by putting

$$\begin{aligned}R_{ij} &= \begin{pmatrix} (e_{j1}, e_{i1}) & \dots & (e_{jn(i)}, e_{i1}) \\ & \dots & \\ (e_{j1}, e_{in(i)}) & \dots & (e_{jn(i)}, e_{in(i)}) \end{pmatrix} \\ &= \begin{pmatrix} e_{i1} Re_{j1} & \dots & e_{i1} Re_{jn(i)} \\ & \dots & \\ e_{in(i)} Re_{j1} & \dots & e_{in(i)} Re_{jn(i)} \end{pmatrix}\end{aligned}$$

Then

$$R = \begin{pmatrix} R_{11} & \dots & R_{1m} \\ & \dots & \\ R_{m1} & \dots & R_{mm} \end{pmatrix}$$

We define $P_{ik,jt}$ corresponding to $e_{ik} Re_{jt}$ as follows :

$$P_{ik,jt} = \begin{cases} e_{i1} Re_{j1} = (e_{j1}, e_{i1}) & (i \neq j) \\ e_{i1} Re_{j1} = (e_{j1}, e_{i1}) & (i = j, k \leq t) \\ J(e_{ij} Re_{j1}) = J(e_{j1}, e_{i1}) & (i = j, k > t) \end{cases}$$

and put

$$P_{ij} = \begin{pmatrix} P_{i1,j1} & \dots & P_{i1,jn(i)} \\ & \dots & \\ P_{in(i),j1} & \dots & P_{in(i),jn(i)} \end{pmatrix}$$

Namely, when $i \neq j$,

$$P_{ij} = \begin{pmatrix} e_{i1} Re_{i1} & \dots & e_{i1} Re_{i1} \\ & \dots & \\ e_{i1} Re_{i1} & \dots & e_{i1} Re_{i1} \end{pmatrix}$$

and when $i = j$,

$$P_{ij} = P_{ii} = \begin{pmatrix} e_{i1} R e_{i1} & \cdots & e_{i1} R e_{i1} \\ & \ddots & \\ J(e_{i1} R e_{i1}) & \cdots & e_{i1} R e_{i1} \end{pmatrix}$$

We put

$$P = \begin{pmatrix} P_{11} & \cdots & P_{1m} \\ & \ddots & \\ P_{m1} & \cdots & P_{mm} \end{pmatrix}$$

Then P becomes a ring by usual matrix operations. Let p_{ij} be the element of P such that its (ij, ij) position is the unity of $P_{ij,ij}$ and all other positions are zero. Then $\{p_{11}, \dots, p_{1n(1)}, \dots, p_{m1}, \dots, p_{mn(m)}\}$ is a complete set of orthogonal primitive idempotents of P ; $P = p_{11}P \oplus \cdots \oplus p_{1n(1)}P \oplus \cdots \oplus p_{m1}P \oplus \cdots \oplus p_{mn(m)}P$.

For each (ik, jt) , we define a mapping

$$\begin{array}{ccc} \tau_{ik,jt}: P_{ik,jt} & \rightarrow & e_{ik} R e_{jt} \\ \cap & & \parallel \\ & & (e_{j1}, e_{i1}) (e_{jt}, e_{ik}) \end{array}$$

by the rule: $\alpha \rightarrow (\theta_{k,1}^i)^{-1} \alpha \theta_{k,1}^j$. Of course, the mapping is well defined. In fact, let $\alpha \in (e_{j1}, e_{i1})$. If $i \neq j$, $\text{Im } \alpha \subseteq J_{n(i)}(e_{i1} R) \subset J_{k-1}(e_{i1} R) = \text{Im } \theta_{k,1}^i$ (Note that R is a basic ring). If $i = j$ and $k \leq t$, then, $\text{Im } \alpha \theta_{k,1}^i \subseteq J_{t-1}(e_{i1} R) \subseteq J_{k-1}(e_{i1} R) = \text{Im } \theta_{k,1}^i$. If $i = j$ and $k > t$, then as $\alpha \in J(e_{i1} R e_{i1})$, α is non unit, so $\text{Im } \alpha \subseteq J_{n(i)}(e_{i1} R) \subseteq J_{k-1}(e_{i1} R) = \text{Im } \theta_{k,1}^i$. Thus $\tau_{ik,jt}$ is well defined. Note that $\tau_{ik,jt}$ is a group homomorphism.

We need the following proposition

Proposition 1. *Each $\tau_{ik,jt}$ is an epimorphism and $\tau_{ik,jt} \tau_{jt,pq} = \tau_{ik,pq}$ for each (ik, jt) and (jt, pq) .*

Proof. Let $\beta \in (e_{jt}, e_{ik})$. When $i \neq j$, consider the following diagram:

$$\begin{array}{ccc} 0 \rightarrow e_{jt} R & \xrightarrow{\theta_{k,1}^j} & J_{t-1}(e_{j1} R) \subseteq e_{j1} R \\ \downarrow \beta & & \\ e_{ik} R & & \\ \downarrow \theta_{k,1}^i & & \\ J_{k-1}(e_{i1} R) & & \\ \cap & & \\ e_{i1} R & & \end{array}$$

Since $e_{i1}R_R$ is injective, there exists $\alpha \in (e_{j1}, e_{i1})$ such that $\alpha\theta_{k,1}^i = \theta_{k,1}^i\beta$. Noting $i \neq j$, we see $\text{Im } \alpha \subseteq J_{n(i)}(e_{i1}R)$; so $\text{Im } \alpha\theta_{k,1}^i \subseteq J_{k-1}(e_{i1}R)$ and hence we see $\beta = (\theta_{k,1}^i)^{-1}\alpha\theta_{k,1}^i = \tau_{ik,jt}(\alpha)$.

When $i = j$ and $k \leq t$, consider the following

$$\begin{array}{c} 0 \rightarrow e_{it}R \xrightarrow{\theta_{k,1}^i} J_{t-1}(e_{i1}R) \subseteq e_{i1}R \\ \downarrow \beta \\ e_{ik}R \\ \downarrow \theta_{k,1}^i \\ J_{k-1}(e_{i1}R) \\ \cap \\ e_{i1}R \end{array}$$

Then there exists $\alpha \in (e_{i1}, e_{i1})$ satisfying $\alpha\theta_{k,1}^i = \theta_{k,1}^i\beta$. Since $k \leq t$, $\text{Im } \alpha\theta_{k,1}^i \subseteq J_{t-1}(e_{i1}R) \subseteq J_{k-1}(e_{i1}R)$. Hence $\beta = (\theta_{k,1}^i)^{-1}\alpha\theta_{k,1}^i = \tau_{ik,ii}(\alpha)$.

Next, when $i = j$ and $k > t$, again consider the diagram

$$\begin{array}{c} 0 \rightarrow e_{it}R \xrightarrow{\theta_{k,1}^i} J_{t-1}(e_{i1}R) \subseteq e_{i1}R \\ \downarrow \beta \\ e_{ik}R \\ \downarrow \theta_{k,1}^i \\ J_{k-1}(e_{i1}R) \\ \cap \\ e_{i1}R \end{array}$$

We take $\alpha \in (e_{i1}, e_{i1})$ satisfying $\alpha\theta_{k,1}^i = \theta_{k,1}^i\beta$. If α is an isomorphism, we see from $k > t$ that $\text{Im } \alpha\theta_{k,1}^i = J_{t-1}(e_{i1}R) \cong J_{k-1}(e_{i1}R) \supseteq \text{Im } \theta_{k,1}^i\beta$, which contradicts $\alpha\theta_{k,1}^i = \theta_{k,1}^i\beta$. Hence α must be non-unit, so $\alpha \in J((e_{i1}, e_{i1}))$. Since $\text{Im } \alpha \subseteq J_{n(i)}(e_{i1}R)$, $\text{Im } \alpha\theta_{k,1}^i \subseteq J_{k-1}(e_{i1}R)$. Hence $\beta = (\theta_{k,1}^i)^{-1}\alpha\theta_{k,1}^i = \tau_{ik,ii}(\alpha)$. Thus we see that $\tau_{ik,jt}$ is an epimorphism.

For $\alpha \in P_{jt,\rho q}$ and $\beta \in P_{ik,jt}$, $\tau_{ik,jt}(\beta)\tau_{jt,\rho q}(\alpha) = ((\theta_{k,1}^i)^{-1}\beta\theta_{k,1}^i)((\theta_{t,1}^j)^{-1}\alpha\theta_{t,1}^j) = (\theta_{k,1}^i)^{-1}\beta\alpha\theta_{k,1}^i$; whence $\tau_{ij,kt}\tau_{jt,\rho q} = \tau_{ik,\rho q}$.

By Proposition 1, we see that

$$\tau = \begin{pmatrix} \tau_{11,11} & \cdots & \tau_{11,mn(m)} \\ & \cdots & \\ \tau_{mn(m),11} & \cdots & \tau_{mn(m),mn(m)} \end{pmatrix}$$

gives a ring epimorphism from P to R with $\tau(p_{ik}) = e_{ik}$ for all ik .

Proposition 2. 1) If $j \neq \sigma(i)$, then $\tau_{ik,jt}$ is an isomorphism.
 2) If $j = \sigma(i)$ and $t \leq \rho(i)$, then $\tau_{ik,jt}$ is also an isomorphism.
 3) If $j = \sigma(i)$ and $t > \rho(i)$, then $\tau_{ik,jt}$ is not an isomorphism. In fact,

$$\begin{aligned} \text{Ker } \tau_{ik,\sigma(i)t} &= \{ \alpha \in P_{ik,\sigma(i)t} \mid \text{Ker } \alpha = J_{\sigma(i)}(e_{\sigma(i)1}R) \} \cup 0 \\ &= S(e_{i1}Re_{i1}e_{i1}Re_{\sigma(i)1}) \\ &= S(P_{ik,ik}P_{ik,\sigma(i)t}) \\ &= S(e_{i1}Re_{\sigma(i)1}e_{\sigma(i)1}Re_{\sigma(i)1}) \\ &= S(P_{ik,\sigma(i)t}P_{\sigma(i)t,\sigma(i)t}) \end{aligned}$$

So, in this case, $\text{Ker } \tau_{ik,jt}$ is a simple module as a left $P_{ik,ik}$ - and a right $P_{j1,jt}$ -module.

Proof. We can take an epimorphism β of $e_{\sigma(i)\sigma(i)}R_R$ to $S(e_{i1}R)_R$.

1) Assume $j \neq \sigma(i)$. If there exists a non-zero $\alpha \in \rho_{ik,jt}$ such that $0 = \tau_{ik,jt}(\alpha) = (\theta_{k,1}^i)^{-1}\alpha\theta_{t,1}^j$, then $\alpha\theta_{t,1}^j = 0$; so $\text{Ker } \alpha \supseteq J_{t-1}(e_{j1}R)$. Consider the diagram

$$\begin{array}{ccc} e_{\sigma(i)\sigma(i)}R & & \\ \downarrow \beta & & \\ S(e_{i1}R) & & \\ \cap & & \\ e_{j1}R \xrightarrow{\alpha} \text{Im } \alpha \rightarrow 0 \end{array}$$

Since $e_{\sigma(i)\sigma(i)}R_R$ is projective, there exists γ such that $\alpha\gamma = \beta$. Since $j \neq \sigma(i)$ and R is basic, γ is not an epimorphism; whence $\text{Im } \gamma \subseteq J_{n(j)}(e_{j1}R) \subseteq J_{t-1}(e_{j1}R) \subseteq \text{Ker } \alpha$ and hence $\beta = \alpha\gamma = 0$, a contradiction. Thus, $\text{Ker } \tau_{ik,jt} = 0$ and hence $\tau_{ik,jt}$ is an isomorphism (cf. Proposition 1).

2) Assume $j = \sigma(i)$ and $t \leq \rho(i)$. If there exists $\alpha \in P_{ik,jt}$ such that $0 = \tau_{ik,\sigma(i)t}(\alpha) = (\theta_{k,1}^i)^{-1}\alpha\theta_{t,1}^{\sigma(i)}$, then $\alpha\theta_{t,1}^{\sigma(i)} = 0$; so $\text{Ker } \alpha \supseteq J_{t-1}(e_{\sigma(i)1}R)$. Consider the diagram:

$$\begin{array}{ccc} e_{\sigma(i)\sigma(i)}R & & \\ \downarrow \beta & & \\ S(e_{i1}R) & & \\ \cap & & \\ e_{\sigma(i)1}R \xrightarrow{\alpha} \text{Im } \alpha \rightarrow 0 \end{array}$$

Since $e_{\sigma(i)\sigma(i)}R_R$ is projective, there exists $\gamma \in (e_{\sigma(i)\sigma(i)}, e_{\sigma(i)1})$ such that $\alpha\gamma = \beta$. If $\rho(i) = 1$, then $t = 1$ since $\rho(i) \geq t$; which implies $\text{Ker } \alpha \supseteq$

$J_0(e_{\sigma(i)}R) = e_{\sigma(i)}R$; so $\alpha = 0$, a contradiction. Hence $\rho(i) \neq 1$. As γ is not an epimorphism (R is basic), $\text{Im } \gamma \subseteq J_{\rho(i)-1}(e_{\sigma(i)}R)$. Since $t \leq \rho(i)$, $J_{\rho(i)-1}(e_{\sigma(i)}R) \subseteq J_t(e_{\sigma(i)}R) \subseteq \text{Ker } \alpha$; whence $\beta = \alpha\gamma = 0$, a contradiction. Therefore $\text{Ker } \tau_{ik, \sigma(i)t} = 0$ and so $\tau_{ik, \sigma(i)t}$ is an isomorphism.

3) Assume $j = \sigma(i)$ and $t > \rho(i)$. We put $X = \{\alpha \in P_{ik, \sigma(i)t} \mid \text{Ker } \alpha = J_{\rho(i)}(e_{\sigma(i)}R)\} \cup 0$ and we first show that $\text{Ker } \tau_{ik, \sigma(i)t} = X$. Let $0 \neq \alpha \in \text{Ker } \tau_{ik, \sigma(i)t}$. Then since $0 = \tau_{ik, \sigma(i)t}(\alpha) = (\theta_{k,1}^t)^{-1} \alpha \theta_{k,1}^{\sigma(i)}$, $\text{Ker } \alpha \supseteq J_{t-1}(e_{\sigma(i)}R)$ (Note that $t \geq 2$ since $t > \rho(i)$). Hence $\text{Ker } \alpha = J_{s-1}(e_{\sigma(i)}R)$ for some $s \leq t$. We claim that $s-1 = \rho(i)$. To see this, let $\bar{\alpha}$ be the restriction of α to $J_{s-2}(e_{\sigma(i)}R)$. Then $\text{Ker } \bar{\alpha} = \text{Ker } \alpha$ and $\text{Im } \bar{\alpha} \simeq J_{s-2}(e_{\sigma(i)}R)/\text{Ker } \bar{\alpha} = J_{s-2}(e_{\sigma(i)}R)/J_{s-1}(e_{\sigma(i)}R)$; so $\text{Im } \bar{\alpha} = S(e_{i1}R)$. Therefore $\bar{\alpha}\theta_{s-1,1}^{\sigma(i)}$ is an epimorphism of $e_{\sigma(i), s-1}R$ to $S(e_{i1}R)$. Hence $s-1 = \rho(i)$ and therefore $\text{Ker } \alpha = J_{\rho(i)}(e_{\sigma(i)}R)$. Thus $\alpha \in X$; whence $\text{Ker } \tau_{ik, \sigma(i)t} \subseteq X$.

Conversely, let $0 \neq \alpha \in X$. Since $\text{Ker } \alpha = J_{\rho(i)}(e_{\sigma(i)}R)$ and $\rho(i) \leq t-1$, $\text{Im } \theta_{k,1}^{\sigma(i)} = J_{t-1}(e_{\sigma(i)}R) \subseteq J_{\rho(i)}(e_{\sigma(i)}R) = \text{Ker } \alpha$; whence $0 = (\theta_{k,1}^t)^{-1} \alpha \theta_{k,1}^{\sigma(i)} = \tau_{ik, \sigma(i)t}(\alpha)$ and $\alpha \in \text{Ker } \tau_{ik, \sigma(i)t}$. Thus $X \subseteq \text{Ker } \tau_{ik, \sigma(i)t}$ and hence $X = \text{Ker } \tau_{ik, \sigma(i)t}$.

Next, we show that $\text{Ker } \tau_{ik, \sigma(i)t} \neq 0$. Consider the diagram :

$$\begin{array}{ccc} 0 \rightarrow e_{\sigma(i)\rho(i)}R & \xrightarrow{\theta_{\rho(i),1}^{\sigma(i)}} & J_{\rho(i)-1}(e_{\sigma(i)}R) \subseteq e_{\sigma(i)}R \\ & \downarrow \beta & \\ & S(e_{i1}R) & \\ & \cap & \\ & e_{i1}R & \end{array}$$

Since $e_{i1}R_R$ is injective, there exists non-zero $\alpha \in P_{ik, \sigma(i)t}$ such that $\beta = \alpha\theta_{\rho(i),1}^{\sigma(i)}$. We claim $\alpha \in \text{Ker } \tau_{ik, \sigma(i)t}$, i.e., $\text{Ker } \alpha = J_{\rho(i)}(e_{\sigma(i)}R)$. If $x \in J_{\rho(i)}(e_{\sigma(i)}R)$, then we see from $(\theta_{\rho(i),1}^{\sigma(i)})^{-1}(x) \in J(e_{\sigma(i)\rho(i)}R) = \text{Ker } \beta$ that $\beta((\theta_{\rho(i),1}^{\sigma(i)})^{-1}(x)) = \alpha\theta_{\rho(i),1}^{\sigma(i)}((\theta_{\rho(i),1}^{\sigma(i)})^{-1}(x)) = \alpha(x) = 0$; so $x \in \text{Ker } \alpha$. Hence $J_{\rho(i)}(e_{\sigma(i)}R) \subseteq \text{Ker } \alpha$. If $\text{Ker } \alpha \supseteq J_f(e_{\sigma(i)}R)$ for some $f < \rho(i)$, then $\beta(e_{\sigma(i)}R) = \alpha\theta_{\rho(i),1}^{\sigma(i)}(e_{\sigma(i)}R) = \alpha(J_{\rho(i)-1}(e_{\sigma(i)}R)) = 0$ and hence $\beta = 0$, a contradiction. Thus we see $\text{Ker } \alpha \subseteq J_{\rho(i)}(e_{\sigma(i)}R)$ and $\text{Ker } \alpha = J_{\rho(i)}(e_{\sigma(i)}R)$.

To show the remainder, we put $\psi = \tau_{i1, \sigma(i)\rho(i)}$; so ψ is an isomorphism of $e_{i1}Re_{\sigma(i)}R$ to $e_{i1}Re_{\sigma(i)\rho(i)}R$ by the rule $\psi(\alpha) = (\theta_{1,1}^t)^{-1} \alpha \theta_{\rho(i),1}^{\sigma(i)} = \alpha\theta_{\rho(i),1}^{\sigma(i)}$, and put $Y = \{\alpha \in (e_{\sigma(i)\rho(i)}, e_{i1}) \mid \text{Im } \alpha \subseteq S(e_{i1}R)\}$. We see from [4] that $0 \neq Y = S(e_{i1}Re_{\sigma(i)\rho(i)}R) = S(e_{i1}Re_{i1}Re_{\sigma(i)\rho(i)}R) = S(e_{i1}Re_{i1}Re_{\sigma(i)}R)$. We show $\psi(X) = Y$. Let $0 \neq \alpha \in X$. If $\text{Ker } \alpha\theta_{\rho(i),1}^{\sigma(i)} = e_{\sigma(i)\rho(i)}R$, then $\psi(\alpha) = \alpha\theta_{\rho(i),1}^{\sigma(i)} = 0$, so $\alpha = 0$, a contradiction. Therefore $\text{Ker } \alpha\theta_{\rho(i),1}^{\sigma(i)} \subseteq J(e_{\sigma(i)\rho(i)}R)$. On the other hand, for

any $x \in J(e_{\sigma(i)\rho(i)}R)$, $\theta_{\rho(i),1}^{\sigma(i)}(x) \in J(J_{\rho(i)-1}(e_{\sigma(i)}R)) = J_{\rho(i)}(e_{\sigma(i)}R) = \text{Ker } \alpha$; so $\alpha\theta_{\rho(i),1}^{\sigma(i)}(x) = 0$ and hence $x \in \text{Ker } \alpha\theta_{\rho(i),1}^{\sigma(i)}$. Thus $J(e_{\sigma(i)\rho(i)}R) \subseteq \text{Ker } \alpha\theta_{\rho(i),1}^{\sigma(i)}$ and hence $J(e_{\sigma(i)\rho(i)}R) = \text{Ker } \alpha\theta_{\rho(i),1}^{\sigma(i)}$, from which we see that $\text{Im } \alpha\theta_{\rho(i),1}^{\sigma(i)} = S(e_{i1}R_R)$ and $\psi(\alpha) \in Y$; as a result, $\psi(X) \subseteq Y$. Conversely, for any non-zero $\alpha \in Y$, we can take $\beta \in e_{i1}Re_{\sigma(i)}R$ such that $\psi(\beta) = \alpha$, since ψ is an isomorphism. If $\text{Ker } \beta \supseteq J_t(e_{\sigma(i)}R)$ for some $t < \rho(i)$, then $\alpha = \psi(\beta) = \beta\theta_{\rho(i),1}^{\sigma(i)} = \beta(J_{\rho(i)-1}(e_{\sigma(i)}R)) = 0$, a contradiction. Hence $\text{Ker } \beta \subseteq J_{\rho(i)}(e_{\sigma(i)}R)$. On the other hand, for any $x \in J_{\rho(i)}(e_{\sigma(i)}R)$, we see from $(\theta_{\rho(i),1}^{\sigma(i)})^{-1}(x) \in J(e_{\sigma(i)\rho(i)}R)$ that $\alpha((\theta_{\rho(i),1}^{\sigma(i)})^{-1}(x)) = 0$. Since $\beta = \alpha\theta_{\rho(i),1}^{\sigma(i)}$, it follows that $\beta(x) = 0$; whence $x \in \text{Ker } \beta$. Thus $\text{Ker } \beta = J_{\rho(i)}(e_{\sigma(i)}R)$ and hence $\beta \in X$ and we obtain $Y \subseteq \psi(X)$.

To show $\psi(X) \subseteq Y$, we put $\mu = \tau_{\sigma(i)\rho(i), \sigma(i)\rho(i)}$, i.e., μ is the mapping of $e_{\sigma(i)}Re_{\sigma(i)}R$ to $e_{\sigma(i)\rho(i)}Re_{\sigma(i)\rho(i)}R$ given by $\mu(\alpha) = (\theta_{\rho(i),1}^{\sigma(i)})^{-1}\alpha\theta_{\rho(i),1}^{\sigma(i)}$. As we saw above, μ is a group isomorphism. Moreover, this map is a ring isomorphism. By this isomorphism, $e_{i1}Re_{\sigma(i)\rho(i)}R$ becomes a right $e_{\sigma(i)}Re_{\sigma(i)}R$ -module. For $\alpha \in e_{i1}Re_{\sigma(i)}R$ and $x \in e_{\sigma(i)}Re_{\sigma(i)}R$, $\psi(\alpha x) = \alpha x \theta_{\rho(i),1}^{\sigma(i)} = (\alpha \theta_{\rho(i),1}^{\sigma(i)}) \cdot ((\theta_{\rho(i),1}^{\sigma(i)})^{-1}x \theta_{\rho(i),1}^{\sigma(i)}) = \psi(\alpha)\mu(x) = \psi(\alpha)x$. Hence ψ is an isomorphism between right $e_{\sigma(i)}Re_{\sigma(i)}R$ -modules. Since $Y = S(e_{i1}Re_{\sigma(i)\rho(i)}e_{\sigma(i)\rho(i)}Re_{\sigma(i)\rho(i)}R)$, it follows that $X = S(e_{i1}Re_{\sigma(i)}e_{\sigma(i)}Re_{\sigma(i)}R) = S(P_{ik, \sigma(i)t}P_{\sigma(i)t, \sigma(i)t})$. Finally, we see from $Y = S(e_{i1}Re_{i1}e_{i1}Re_{\sigma(i)\rho(i)}R)$ and $e_{i1}Re_{i1}e_{i1}Re_{i1} \simeq e_{i1}Re_{i1}e_{i1}Re_{\sigma(i)}R$ that $\text{Ker } \tau_{ik, \sigma(i)t} = S(P_{ik, ik}P_{ik, \sigma(i)t})$.

We replace $P_{ik, \sigma(i)t}$ in

$$P_{i\sigma(i)} = \begin{pmatrix} P_{i1, \sigma(i)} & \cdots & P_{i1, \sigma(i)n(\sigma(i))} \\ & \cdots & \\ P_{in(i), \sigma(i)} & \cdots & P_{in(i), \sigma(i)n(\sigma(i))} \end{pmatrix}$$

by $P_{ik, \sigma(i)t}/S(P_{ik, \sigma(i)t})$ for $k = 1, \dots, n(i)$ and $j = \rho(i)+1, \dots, n(\sigma(i))$, and denote it by $P_{ik, \sigma(i)}^*$. And we put

$$R^* = \begin{pmatrix} P_{11} \cdots P_{1, \sigma(i)-1} P_{1\sigma(i)}^* P_{1, \sigma(i)+1} \cdots P_{1m} \\ \cdots \\ P_{m1} \cdots P_{m, \sigma(m)-1} P_{m\sigma(m)}^* P_{m, \sigma(m)+1} \cdots P_{mn} \end{pmatrix}$$

Then by Propositions 1 and 2, we obtain

Theorem 1. R^* canonically becomes a ring such that $R^* \simeq P/\text{Ker } \tau \simeq R$. Therefore R^* is a representative matrix ring of R .

We shall illustrate this theorem for two cases :

1) The case : $m = 1$. Then

$$R \simeq R^* = \begin{pmatrix} \overbrace{Q \cdots Q \bar{Q} \cdots \bar{Q}}^{\rho(1)} \\ J \ddots & : & : & : \\ : & \ddots & Q & : \\ : & & J \bar{Q} & : \\ : & & : \bar{J} & : \\ : & & : & \ddots \\ J \cdots & J & \bar{J} \cdots & \bar{J} \bar{Q} \end{pmatrix}$$

where $Q = e_{11}Re_{11}$, $J = J(Q)$, $\bar{Q} = Q/S(Q)$. (In this case, Q is a local QF-ring, since both ${}_Q S({}_Q Q)$ and $S(Q_Q)_Q$ are simple (cf. [5])).

2) The case : $m = 2$, $\sigma(1) = 2$ and $\sigma(2) = 1$. Then

$$R \simeq R^* = \begin{pmatrix} \overbrace{Q \cdots Q A \cdots A \bar{A} \cdots \bar{A}}^{\rho(1)} \\ : & : & : & : \\ J & & Q A \cdots A \bar{A} \cdots \bar{A} \\ B \cdots B \bar{B} \cdots \bar{B} T \cdots T \\ : & : & : & : \\ : & : & : & : \\ \underbrace{B \cdots B \bar{B} \cdots \bar{B}}_{\rho(2)} & K & & T \end{pmatrix}$$

where $Q = e_{11}Re_{11}$, $T = e_{21}Re_{21}$, $A = e_{11}Re_{21}$, $\bar{A} = A/S(A)$, $B = e_{21}Re_{11}$,

$\bar{B} = B/S(B)$, $J = J(Q)$ and $K = J(T)$. (Then $T = \begin{pmatrix} Q & A \\ B & T \end{pmatrix}$ is a QF-ring,

since $(eT_\tau; {}_\tau T f)$ and $(fT_\tau; {}_\tau T e)$ are injective pairs, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$)

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